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Integrable geodesic flows on n-step nilmanifolds

Leo Butler

Department of Mathematics and Statistics, Queen's University, Kingston, Ont., Canada K7L 3N6

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Abstract

Recent examples of Liouville-integrable geodesic flows on non-simply connected manifolds have shown that the topological implications of C^∞ Liouville integrability are dramatically different from the implications of real-analytic integrability. In particular a geodesic flow can be both smoothly integrable and have positive topological entropy [A.V. Bolsinov, I.A. Taĭmanov, Russ. Math. Surveys 54 (4) (1999) 833–835]. The examples of Bolsinov and Taĭmanov, and of Butler [L. Butler, CR Math. Rep. Acad. Sci. Can. 21 (4) (1999) 127–131] are constructed from left-invariant metrics on Lie groups. In this paper, the degeneracy of the Poisson tensor on the dual algebra is shown to be the source of the large number of commuting first integrals, and additional examples of integrable geodesic flows are constructed on n-step nilmanifolds.

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1. Introduction

A Riemannian metric on a smooth manifold M^n induces a geodesic flow ϕ_t on T^*M^n, a flow that is Hamiltonian. It is very rare for this flow to be Liouville integrable, and one would like to know: what are the topological implications of Liouville integrability? Taĭmanov has shown that if M^n is real analytic and all first integrals of ϕ_t are real analytic, then there are strong restrictions on the topology of M^n: its fundamental group must be almost Abelian and its rational cohomology ring must contain a subring isomorphic to $H^*(\mathbb{T}^d;\mathbb{Q})$ where $d = \dim H^1(M;\mathbb{Q})$ [11,12].

For geodesic flows that are Liouville integrable with smooth (C^∞) first integrals, the topological implications are much weaker. Paternain has proven a number of results in this direction, and in each case the hypotheses are modeled on the behavior of real-analytically

E-mail address: lbutler@mast.queensu.ca (L. Butler).

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Liouville-integrable systems ([8–10], see also [5]). In [2,4], examples are presented of real-analytic geodesic flows that are smoothly integrable and whose first integrals satisfy none of the hypotheses of [5,8–10]. The examples are obtained from compact quotients of two-step nilpotent and two-step solvable Lie groups, respectively. One might wonder: are there smoothly integrable geodesic flows on manifolds with n-step nilpotent (resp. solvable) fundamental groups?

In this note it is shown that the geodesic flow of a left-invariant metric on a family of n-step nilpotent Lie groups induces Liouville-integrable geodesic flows on compact quotients. This paper also shows that the highly degenerate nature of the Poisson tensor on the Lie coalgebra accounts for the Liouville integrability of the geodesic flows here and in [2,4].

1.1. A statement of the results

The following main theorems are proven.

Theorem 1.1 (Integrability of the geodesic flow). *Let $n \geq 2$ and $N \in gl(n; \mathbb{R})$ be nilpotent: $N^k = 0$ for some $k \geq 0$. Define the multiplication $*$ on $G = G_N = \mathbb{R} \times \mathbb{R}^n$ by*

$$(x, y) * (x', y') := (x + x', y + \exp(xN)y'),$$

where exp *is the usual matrix exponential function. Let $D \leq G$ be a lattice and g be a left-invariant metric on G; then the geodesic flow of $H(p, q) = \frac{1}{2}g^{-1}(p, p)$ is Liouville integrable on $T^*(D \setminus G)$ with n real-analytic first integrals and a single C^∞ first integral*.

Remark 1.2. (i) *The n first integrals are even algebraic, not simply real-analytic.* (ii) *A lattice in a simply connected Lie group is a discrete, cocompact subgroup. $G = G_N$ possesses a lattice D iff there is a basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ and an $x \in \mathbb{R}$ such that for each i,* $\exp(xN)v_i$ *is in the $\mathbb{Z}$-span of $v_1, \dots, v_n$. That is, without loss of generality, one may assume that* $\exp(N) \in \mathrm{SL}(n; \mathbb{Z})$ *relative to the standard basis of $\mathbb{R}^n$, and that the coordinates of each element in the lattice D are integers*.

Paternain [8–10] has proven a number of results concerning the topology of manifolds which possess geodesic flows with zero topological entropy. This family of examples fits within that class:

Theorem 1.3. *Let G_N, D, g, H be as above. Then the geodesic flow of g on the unit cotangent bundle $S^*(D \setminus G)$ has zero topological entropy*.

Remark 1.4. (i) *There is a published proof due to Manning* [7] *that the topological entropy of a left-invariant geodesic flow on a nilmanifold must vanish. This proof is mistaken: it assumes that the exponential map of the metric is the same as the exponential map of the group, which requires bi-invariance of the metric. This means that the Lie algebra admits a positive-definite ad-invariant quadratic form so the Lie algebra must be reductive. The only connected, simply connected Lie groups that are both nilpotent and reductive are* $(\mathbb{R}^n, +)$, *so Manning's proof works only for* $\mathbb{T}^n$. *It remains an open question if the topological entropy of a left-invariant geodesic flow on a nilmanifold is always zero.* (ii) *Bolsinov and Taĭmanov* [2] *exhibit Liouville-integrable geodesic flows on a three-dimensional solvmanifold* M^3 *with positive topological entropy. This paper provides an interpretation of their example: it is a left-invariant metric on a Lie group* G_N *with N semisimple and* $M^3 = D \setminus G$. *The integrability of the geodesic flow in their example arises because of the extreme degeneracy of the Poisson tensor on* $\mathcal{G}_N^*$ (*see Lemma* 1.7 *and Remark* 2.5).

Remark 1.5. *There is a special case where* $\mathbb{R}^n$ *has a basis* $e_1, \dots, e_n$ *such that*

$$Ne_i = \begin{cases} e_{i-1}, & n \geq i \geq 2, \\ 0, & i = 1. \end{cases}$$

In this case, there is a basis $X, Y_1, \dots, Y_n$ *of the Lie algebra* $\mathcal{G}$ *of G such that* $[Y_i, Y_j] = 0$ *for* $1 \leq i, j \leq n$ *and*

$$[X, Y_i] = \begin{cases} Y_{i-1}, & n \geq i \geq 2, \\ 0, & i = 1. \end{cases} \tag{1}$$

The lower central series of $\mathcal{G}$ *is then* $\mathcal{G}_0 = \mathcal{G}$, *and for* $n-1 \geq k \geq 1$, $\mathcal{G}_k = [\mathcal{G}, \mathcal{G}_{k-1}] = \operatorname{span}\{Y_{n-k}, \dots, Y_1\}$. *Hence, the Lie algebra of G, and so G, is n-step nilpotent with* $\dim \mathcal{G}_k/\mathcal{G}_{k+1} = 2$ *if* $k = 0$, *and* 1 *otherwise.*

Corollary 1.6. *Let N be the* $n \times n$ *matrix*

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

and define the Lie group $G = G_N$ *as in Theorem* 1.1. *Let* $D \leq G$ *be a lattice in G. Then* $\pi_1(D \setminus G) \simeq D$ *has word growth of degree* $1 + \frac{1}{2}n(n+1)$, *and* $D \setminus G$ *admits a Liouville-integrable geodesic flow.*

This corollary is a simple application of a theorem due to Bass [1] that the word growth of a finitely generated nilpotent group D with lower central series $D = D_0 \geq D_1 \geq \cdots \geq D_{c-1} \geq 1 = D_c$, $D_{k+1} = [D, D_k]$ is a polynomial with degree d equal to the sum

$$d = \sum_{k=1}^{c} k \operatorname{rank}\left(\frac{D_{k-1}}{D_k}\right).$$

It is clear that for D of fixed rank r (any finitely-generated, torsion-free nilpotent group can be embedded as a discrete, cocompact subgroup in a simply connected nilpotent Lie group; the rank of the group is the dimension of the Lie group [6]) its nilpotency class $c \leq r-1$ and the word-growth function has degree $d \leq 1 + \frac{1}{2}r(r-1)$. This upper bound on the degree of the word-growth function is achieved by the lattices in the above example. It should now be apparent that, if the algebraic structure of the fundamental group is an obstruction to the existence of Liouville-integrable geodesic flows, then this structure is not captured by an invariant like the word-growth function.

The proof of the Liouville integrability of the geodesic flow is obtained from the following lemma:

Lemma 1.7. *Let G_N be defined for $N \neq 0 \in gl(n;\mathbb{R})$. Then the Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{G}^*$ satisfies*

$$\{p_\alpha, p_{\beta_i}\} = -(N' p_\beta)_i, \tag{2}$$

for $i = 1, \ldots, n$ and all other brackets are zero. Hence, the Poisson tensor is

$$\mathcal{P} = \left(\sum_{i=1}^{n} (N' p_\beta)_i \frac{\partial}{\partial p_{\beta_i}}\right) \wedge \frac{\partial}{\partial p_\alpha}, \tag{3}$$

and it generically has rank 2.

Here and henceforth, $\mathcal{G}$ will be given a basis $X, Y_1, \ldots, Y_n$ such that $[X, Y_i] = NY_i$ and $[Y_i, Y_j] = 0$ for all $1 \leq i, j \leq n$, and for all $p \in \mathcal{G}^*\, p_\alpha = p(X)$, $p_{\beta_i} = p(Y_i)$ and N' is the transpose of N. The pairing of a vector $v \in \mathcal{G}$ and $p \in \mathcal{G}^*$ will be denoted by $p(v)$ and $\langle p, v\rangle$.

It should be remarked that the lemma does not use the nilpotency of N, so it is true for all N, and therefore solvable (but not necessarily nilpotent) G_N.

Corollary 1.8. *Let G_N be defined for $N \in sl(n;\mathbb{R})$, with N nilpotent. Then there are $n-2$ functionally independent first integrals (Casimirs) of the Poisson tensor $\mathcal{P}$ on $\mathcal{G}^*$.*

If $D \leq G$ is a lattice, then these first integrals descend to $T^*(D \setminus G) = D \setminus G \times \mathcal{G}^*$ as Poisson commuting first integrals of any left-invariant Hamiltonian $H : T^*G \to \mathbb{R}$.

2. Proofs

Proof of Lemma 1.7. Let $N \in gl(n;\mathbb{R})$ and define $G = G_N := \mathbb{R} \times \mathbb{R}^n$, where

$$(x, y) * (x', y') := (x + x', \exp(xN)y' + y).$$

Let $\mathcal{G}$ be the Lie algebra of left-invariant vector fields on G, and write $\mathcal{G} = A \oplus B$, where B is the Lie algebra of the normal, closed, Abelian subgroup $0 \times \mathbb{R}^n$ of G, and A a complementary subspace which is identified with $\mathbb{R}$. Let $X, X' \in A$; $Y, Y' \in B$ so that

$$[X + Y, X' + Y'] = XNY' - X'NY \tag{4}$$

is the Lie bracket. $\mathcal{G}$ has a basis $X, Y_1, \ldots, Y_n$ with X a basis of A and $Y_1, \ldots, Y_n$ a basis of B. The Lie coalgebra $\mathcal{G}^*$ is identified with $A^* \oplus B^*$ so that $\mathcal{G}^* \ni p = p_\alpha + p_\beta \in A^* \oplus B^*$. Then the Poisson bracket can be written as

$$\{p_\alpha, p_{\beta_i}\}(p) = -\langle p, [X, Y_i]\rangle = -\langle p_\beta, NY_i\rangle = -(N' p_\beta)_i,$$

where N' is the transpose of N. The brackets $\{p_{\beta_i}, p_{\beta_j}\} = 0$ for all i, j.

The Poisson tensor $\mathcal{P}$ associated with $\{\cdot, \cdot\}$ is

$$\mathcal{P} = -U \wedge V, \tag{5}$$

where $U = \partial/\partial p_\alpha$ and $V = \sum_{i=1}^{n} (N' p_\beta)_i (\partial/\partial p_{\beta_i})$. $\mathcal{P}$ has rank 2 for all $p_\beta \notin \ker N'$, which is an open dense set provided $N \neq 0$. □

Proof of Lemma 1.8. A Casimir f of $\mathcal{P}$ is a smooth function such that $\{\cdot, f\} = \mathcal{P}\,\mathrm{d}f \equiv 0$. If $f = f(p_\alpha, p_\beta)$ is a Casimir, then

$$0 = \mathcal{P}\,\mathrm{d}f = -U(f)V + V(f)U,$$

so that f must be a first integral of both U and V. So $f = f(p_\beta)$ and f is a first integral of the linear differential equation:

$$\dot{p}_\beta = N' p_\beta. \tag{6}$$

To find the first integrals of V (6), let $\mathbb{R}^{n*}$ split into N'-invariant, irreducible subspaces E_i for $i = 1, \ldots, k$, $\dim E_i = n_i$. By the Jordan canonical-form theorem, there is a basis $\mathcal{B}_i$ of each subspace such that $N'|_{E_i}$ relative to this basis has the simple form:

$$N'|_{\mathcal{B}_i} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

First integrals of $V|_{E_i}$ are first integrals of

$$Y_n = \{\dot{y}_1 = 0, \dot{y}_2 = y_1, \ldots, \dot{y}_n = y_{n-1}, \tag{7}$$

where coordinates y_i are relative to the basis $\mathcal{B}_i$. Let Y_n denote this vector field. A first integral of Y_n is a first integral of Y_m for all $m \geq n$. A complete set of first integrals of a vector field is a maximal family of almost everywhere functionally independent first integrals.

Lemma 2.1. *For $n \geq 3$, define*

$$a_{1,n} = -a_{2,n} = n - \tfrac{1}{2}, \qquad a_{j,n} = (-1)^{j+1}(n + \tfrac{1}{2} - j) \quad \text{for} \quad 3 \leq j \leq n-1,$$
$$b_{j,n} = (-1)^j \quad \text{for} \quad 2 \leq j \leq n-1, \qquad b_{n,n} = -a_{n,n} = \tfrac{1}{2}(-1)^n.$$

Let $b_{2,2} = \frac{1}{3}$, $a_{1,2} = -a_{2,2} = 1$. For $n \geq 2$ define the polynomials

$$f_n(y) = (-1)^n \tfrac{1}{2} y_n^2 + \sum_{i=1}^{n-1} (-1)^i y_i y_{2n-i}, \tag{8}$$

and

$$g_n(y) = \sum_{i=1}^{n} a_{i,n} y_1 y_i y_{2n+1-i} + \sum_{j=2}^{n} b_{j,n} y_2 y_j y_{2n-j} \tag{9}$$

and let $f_1(y) = y_1$. *Then* $f_1, \dots, f_n, g_2, \dots, g_{n-1}$ *is a complete family of first integrals for* Y_{2n-1} *and* $f_1, \dots, f_n, g_2, \dots, g_n$ *is a complete family of first integrals for* Y_{2n}.

Proof. A calculation. □

In the general case where $\mathbb{R}^{n*} = E_1 \oplus \cdots \oplus E_k$ is the direct sum of N'-irreducible subspaces, this computation gives $n - \sum_{i:n_i \geq 2} 1$ first integrals of the vector field $V\colon \dot{p} = N'p$. However, the following point should be noted.

Lemma 2.2. *If* $y_{i,j}$ *are the coordinates of* $y \in \mathbb{R}^{n*}$ *relative to the bases* $\mathcal{B}_i$ *of the subspaces* E_i *and* $n_i = \dim E_i \geq 2$ *then* $h_i(y) = y_{1,1} y_{i,2} - y_{1,2} y_{i,1}$ *is a first integral of V for* $i \geq 2$.

Let $f_{i,a}$ ($g_{i,b}$) be the polynomial function f_a (g_b) defined on the subspace E_i. The collection of first integrals $f_{i,a}, g_{i,b}, h_c$ therefore gives a complete ($n - 1$ in number) set of functionally independent first integrals for the vector field $V\colon \dot{p} = N'p$. This completes the proof of Lemma 1.8.

The dimension of T^*G is $2n + 2$, and so far n first integrals for a left-invariant geodesic flow on T^*G have been produced. It remains to produce one additional, independent first integral of the geodesic flow that Poisson commutes with the first integrals of the Poisson structure $\mathcal{P}$ on $\mathcal{G}^*$ exhibited above.

Lemma 2.3. *Let* $(u, v) \in \mathbb{R} \times \mathbb{R}^n \simeq T_1 G$ *and let* $\mathcal{X}(u, v)$ *be the right-invariant extension of* (u, v) *to G. Then, the Hamiltonian of the cotangent lift of* $\mathcal{X}(u, v)$ *to* $T^*G = G \times \mathcal{G}^*$ *is given by*

$$H_{(u,v)}(x, y, p_\alpha, p_\beta) = p_\alpha u + \mathrm{e}^{-xN'} p_\beta (v + uNy). \tag{10}$$

Proof. The right-invariant extension of (u, v) is $\mathcal{X}(u, v)(x, y) = (u, uNy + v)$ and the left-invariant extension of $(p_\alpha, p_\beta) \in T_1^* G$ is $(p_\alpha, p_\beta)(x, y) = (p_\alpha, \mathrm{e}^{-xN'} p_\beta)$. □

Lemma 2.4. *Let* $v \in \mathbb{R}^n$ *be such that* $Nv \neq 0$, $N^2 v = 0$. *Let* $\phi(x) = \exp(-1/x^2)$ *for all* $x \in \mathbb{R}$ *and define the function*

$$f(x, p_\beta) := \phi(p_\beta(Nv)) \sin 2\pi \left[\frac{\mathrm{e}^{-xN'} p_\beta(v)}{p_\beta(Nv)} \right]. \tag{11}$$

Then f is a C^∞ *function on* T^*G *that is invariant under the action of any discrete subgroup* $D \subset \mathbb{Z} \times \mathbb{Z}^n$ *of G.*

Proof. Such a v exists because $0 \neq N$ is nilpotent. Because $N^2 v = 0$, $\mathrm{e}^{-xN} v = v - xNv$ and $\mathrm{e}^{-xN} Nv = Nv$ for all $x \in \mathbb{R}$. From (10), the Hamiltonian of the cotangent lifts of

$\mathcal{X}(0, v)$ and $\mathcal{X}(0, Nv)$ are

$$H_v = p_\beta(v) - xp_\beta(Nv), \qquad H_{Nv} = p_\beta(Nv).$$

That is, $H_v = p_\beta(v) - xH_{Nv}$ so the ratio H_v/H_{Nv} mod 1 is invariant under the translation $x \to x + 1$, when the denominator is non-zero. The definition of f takes care of this singularity. □

Proof of Theorem (1.1) (conclusion). The Hamiltonians H_v and H_{Nv} are Hamiltonians of cotangent lifts of right-invariant vector fields, so they Poisson commute with all left-invariant functions. Functional independence of the first integrals $f, f_{i,a}, g_{j,b}, h_c$ is obvious. To complete the proof that all left-invariant metrics induce Liouville-integrable geodesic flows on all compact quotients one notes that if D is a cocompact, discrete subgroup of G, then there exists an isomorphism $\phi : D' \to D$, where $D' \subset \mathbb{Z}\times\mathbb{Z}^n$ and the isomorphism ϕ of discrete, torsion-free subgroups extends to an isomorphism of connected, simply connected nilpotent Lie groups $\phi : G' \to G$ [6]. The isomorphism ϕ then extends to a symplectomorphism of $T^*G' \to T^*G$, and $\phi_*\mathcal{P}' = \mathcal{P}$, where $\mathcal{P}'$ ($\mathcal{P}$) is the Poisson tensor on $\mathcal{G}'^*$ ($\mathcal{G}^*$). Consequently, if H is a left-invariant function on $T^*(D \setminus G)$ then ϕ^*H is a left-invariant function on $T^*(D' \setminus G')$, which is Liouville integrable by Lemmas 2.1 and 2.4 and so H is Liouville integrable. Letting H be the Hamiltonian of a left-invariant metric proves the theorem. □

Remark 2.5 (cf. [2]). *If $N \in sl(n;\mathbb{R})$, $\exp(N) \in SL(n;\mathbb{Z})$ is semisimple with non-zero real eigenvalues $\mu_1, \dots, \mu_n$ and corresponding eigenvectors $v_1, \dots, v_n$, then the Casimirs of $\mathcal{P}$ are generated by the functions $F = \Pi_{i=1}^n p_\beta(v_i)$, and $f_{ij} = \phi(F)(\mu_i \ln|p_\beta(v_j)| - \mu_j \ln|p_\beta(v_i)|)$, where $\phi(x) = \exp(-1/x^2)$. An additional first integral of a left-invariant metric arises from $h_j = \phi(F)\sin 2\pi(\ln|p_\beta(v_j)| - x)$. Any left-invariant Hamiltonian on $T^*(D \setminus G)$ is therefore Liouville integrable.*

For the metric Hamiltonian $2H = p_\alpha^2 + |p_\beta|^2$ a computation reveals that the time-1 map of its flow along the invariant set ($p_\alpha = 1$, $p_\beta = 0$) is the mapping $(D(x, y), 1, 0) \to (D(x, \exp(-N)y), 1, 0)$, which is an Anosov mapping of the torus $x \equiv$ constant, $p_\alpha = 1$, $p_\beta = 0$ to itself. Hence, the topological entropy is positive.

2.1. Topological entropy

Proof of Theorem 1.3. The theorem will be proven by induction on the dimension n. For $n = 1$, the theorem is clearly true because the matrix $N = 0$ and the group $G_N = \mathbb{T}^2$ with the Euclidean metric on it.

Assume therefore that the topological entropy of all geodesic flows of the type (G_N, D, g, H) vanishes for dim dom $N = 1, \dots, n - 1$. Let dim dom $N = n$ with N satisfying the hypotheses of the theorem, and (G_N, D, g, H) be a 4-tuple of group, lattice, left-invariant metric and Hamiltonian. The aim is to show that the topological entropy of the geodesic flow of g on $S^*(D \setminus G) = H^{-1}(\frac{1}{2})$ vanishes, which is most easily done by showing that it vanishes on $T^*(D \setminus G)$. Let $Z_1, \dots, Z_k$ be a basis of the center of $\mathcal{G}$ and assume that each vector field generates a $\mathbb{T}^1$ action on $T^*(D \setminus G)$.

Note that the linear first integrals of the Poisson tensor (Lemma 2.1) are just linear combinations of the first integrals $p(Z_1), \ldots, p(Z_k)$. Without loss of generality, it may be supposed that the linear first integrals are exactly $p(Z_1), \ldots, p(Z_k)$, and that the first integral f in (11) is chosen so that $Nv = Z_i$ for any $i = 1, \ldots, k$. It follows, therefore, that the first integrals of the geodesic flow are functionally independent of the set $R := \{(Dx, p) \in T^*(D \setminus G) : \Pi_{i=1}^k p(Z_i) \neq 0\}$. The geodesic flow on R is conjugate to a translation-type flow and so $h(\varphi_t|R) = 0$. It remains to examine the flow on R^c.

The set $R^c = \cup_{i=1}^k S_i$, where $S_i := \{(Dx, p) \in T^*(D \setminus G) : p(Z_i) = 0\}$. Each subset (indeed, submanifold) S_i is invariant under the geodesic flow. It will be shown that $h(\varphi_t|S_i) = 0$ for all i.

Let $S = S_i$ and S^1 denote the group whose infinitesimal generator on $T^*(D\setminus G)$ is $X_{p(Z_i)}$. Then, S/S^1 is the symplectic reduction of the zero momentum level set of the momentum map $p(Z_i)$. As remarked above, S is a submanifold because the momentum map is linear: its differential does not vanish on S. Because the geodesic flow is invariant under this S^1 action, the flow φ_t and the Hamiltonian H descend to S.

Claim. *Let $D' := D/D \cap S^1$, $G' := G/S^1$. Then $S = T^*(D' \setminus G')$, the reduced symplectic form coincides with the canonical symplectic form on the cotangent bundle of G': $\omega_S = \omega_{\mathrm{can}}$; and the reduced Hamiltonian H_S coincides with the Hamiltonian of a left-invariant metric on G'.*

Check. The submanifold $S = D\setminus G \times \ker Z_i$, where Z_i is identified as a linear functional on $\mathcal{G}^*$. By virtue of the fact that the action generated by Z_i commutes with that of D, its action on $\mathcal{G}^*$ is trivial so the second isomorphism theorem implies, $S/S^1 = (S^1 D \setminus G) \times \ker Z_i \simeq D' \setminus G' \times \ker Z_i$ and $\ker Z_i$ is naturally identified as $\mathcal{G}'^*$.

The remaining two claims are clear.

From this claim, it follows that the geodesic flow on S, when quotiented by the action of the compact symmetry group S^1 is again a geodesic flow of a left-invariant metric on a manifold in the class $({G'}_{N'}, D', g', H')$, where N' is the nilpotent linear transformation on $\mathcal{G}' = \mathcal{G}/Z_i$ that is induced by the linear transformation N. This manifold has dimension $n - 1$ and so the induction hypothesis applies: $h({\varphi'}_t|T * (D' \setminus G')) = 0$, where ${\varphi'}_t$ is the geodesic flow of g'.

Bowen's theorem [3] implies that the topological entropy of φ_t vanishes on S: $h(\varphi_t|S) = 0$. The supremum rule for topological entropy, $h(\varphi_t) = \sup\{h(\varphi_t|X_i)|\cup X_i = X, X_i \text{ is } \varphi_t\text{–invariant}\}$, then implies that the topological entropy of φ_t vanishes:

$$h(\varphi_t|T^*(D \setminus G)) = 0.$$

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